A New Numerical Method for Synthesis of Arbitrarily Terminated Lossless Nonuniform Transmission Lines

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***Abstract*—In this paper, the synthesis of a nonuniform transmission line is treated by solving an inverse classical Sturm–Liouville problem, in which the boundary conditions are described by S-parameters. The related inverse problem is readily solvable if the terminated impedances and S-parameters satisfy some required conditions. This method can be used to design transmission-line filters and impedance transducers for almost arbitrarily provided source and load impedances.**

***Index Terms*—Filter, inverse problem, nonuniform transmission line, synthesis.**

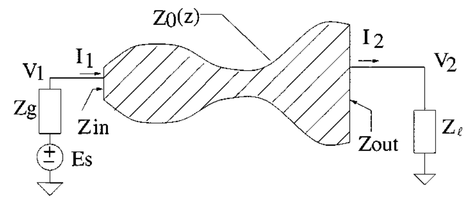

Fig. 1. NTL with characteristic impedance $Z_0(z)$, source impedance Z_g, and load Z_ℓ.

I. Introduction

MANY authors have contributed to the study on nonuniform transmission lines (NTLs) in both frequency and time domain [1]–[3], though most of them are mainly concerned with the analysis problem. Some authors have investigated the synthesis problem [4], [5], treating the inverse scattering problem in time domain, aiming at extending the function of time-domain reflectometry (TDR) in order that TDR can be used to reconstruct NTLs. This kind of method is also applied to time-domain pulse formation [6]. A related technique has been developed by Huang *et al.* [7] for designing quasi-transversal filters. It is efficient for chirp filter designs, especially when superconducting microstrip lines are used. Clearly, this method is not aimed at matching a specific load impedance Z_ℓ.

Wohlers [8] described a synthesis approach in frequency domain, after having examined the realization of an NTL. He delicately transformed the synthesis problem to an inverse Sturm–Liouville problem (SLP), adopting Marchenko' algorithm, but the approach seems unsuitable for engineering applications due to its low computational efficiency. A more efficient method is to solve the Zakharov–Shabat (ZS)-type inverse scattering problem from a given reflection coefficient [9], [10]. This method is confined to filter designing. It ignored the boundary conditions for an NTL at the load, simply truncating the potential function $q(x)$ in order that $q(x)$ is zero when $x < 0$ or $x > \ell$. However, truncating $q(x)$ is different from truncating the time-domain response, and those papers gave no detailed analysis on the effect of truncating $q(x)$. Roy *et al.* presented a simpler numerical approach [11], where characteristic impedance $Z_0(x)$ is interpolated with cubic functions between some selected points, and the design accuracy can be easily controlled.

This paper presents a new method for NTL synthesizing in the frequency domain, also based on the Telegrapher's equations that describe continuously varying NTLs. The boundary conditions are explicitly described by S-parameters. The synthesis of NTLs is changed into solving the related inverse classical Sturm–Liouville's problem. It will be seen that this kind of inverse problem can be carried out easily when S-parameters are provided. Section II provides some preliminary descriptions on this problem. A kind of inverse SLP, which is different from the ZS type and that proposed by Wohlers, is proposed to synthesize NTLs in Section III. A general numerical procedure for synthesizing NTLs is available in Section IV. Especially in Section IV, a practical algorithm and examples for real Z_ℓ and Z_g are provided. A device is designed and constructed based on this algorithm. The simulated and measured results verify that this method is effective to synthesize NTLs from S-parameters for specific Z_ℓ and source impedance Z_g. The method also provides a practical means to adjust local frequency response of an NTL to a certain extent.

II. Preliminaries

In this section, we give a short review over some elementary results of NTL synthesizing that will help us to illustrate our numerical method. We start from the following equations describing the current $I(z,\,\omega)$ and voltage $V(z,\omega)$ in an NTL (see Fig. 1):

$$\frac{dV}{dz} = -j\omega L(z)I \tag{1}$$

$$\frac{dI}{dz} = -j\omega C(z)V. \tag{2}$$

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Following the process in [1], the electrical position and the local characteristic impedance are defined by

$$x(z) = \int_0^z \sqrt{L(s)C(s)}\, ds \tag{3}$$

and

$$Z_0(x) = \sqrt{\frac{L[z(x)]}{C[z(x)]}} \tag{4}$$

respectively. With them, (1) and (2) are transformed into

$$\frac{dV}{dx} = -j\omega Z_0(x) I \tag{5}$$

$$\frac{dI}{dx} = -\frac{j\omega}{Z_0(x)} V. \tag{6}$$

Letting $\phi(x, \omega) = V(x, \omega)/\sqrt{Z_0(x)}$ and $\psi(x, \omega) = I(x, \omega)\sqrt{Z_0(x)}$, we have the following equations:

$$\frac{d\phi(x)}{dx} - k(x)\phi(x) + j\omega\psi(x) = 0 \tag{7}$$

$$\frac{d\psi(x)}{dx} + k(x)\psi(x) + j\omega\phi(x) = 0 \tag{8}$$

and

$$\phi''(x) + [\lambda - q(x)]\phi(x) = 0 \tag{9}$$

$$\psi''(x) + [\lambda - q_1(x)]\psi(x) = 0 \tag{10}$$

where $k(x)$ and potentials $q(x)$ and $q_1(x)$ satisfy

$$\left(\frac{1}{\sqrt{Z_0(x)}}\right)'' - q(x)\frac{1}{\sqrt{Z_0(x)}} = 0 \tag{11}$$

$$\left[\sqrt{Z_0(x)}\right]'' - q_1(x)\sqrt{Z_0(x)} = 0 \tag{12}$$

$$\left(\frac{1}{\sqrt{Z_0(x)}}\right)' - k(x)\frac{1}{\sqrt{Z_0(x)}} = 0 \tag{13}$$

and $\lambda = \omega^2$. Equations (9) and (10) are classical Sturm–Liouville equations. In a synthesis problem, we usually form prerequisite boundary conditions for (9) or (10), and then solve the inverse classical SLP to obtain $q(x)$ or $q_1(x)$. Finally, we determine a practical NTL fabrication using (11) or (12).

For the convenience of further discussion, we quote an important theorem related to this kind of inverse problem [13]:

Theorem 1: Let us write two sets of boundary conditions of (9) as

$$\phi'(0) - h\phi(0) = 0 \tag{14}$$

$$\phi'(\ell) + H_1\phi(\ell) = 0 \tag{15}$$

$$\phi'(\ell) + H_2\phi(\ell) = 0. \tag{16}$$

Let μ_j denote the eigenvalue sequence of (9) with (14) and (15), and λ_j that of (9) with (14) and (16), then when $H_1 \neq H_2$, these two sequences determine $q(x)$, H_1, H_2, and h uniquely.

This theorem forms the foundation of our synthesis theory in the following section.

III. Synthesis of an NTL from Its S-Parameters

In Fig. 1, Z_g and Z_ℓ are arbitrarily given source and load impedances, respectively. ℓ is the electrical length of the NTL. The incident and reflected waves are defined by [1]

$$2h_g a_1 = V_1(j\omega) + Z_g I_1(j\omega) \tag{17}$$

$$2h_g^* b_1 = V_1(j\omega) - Z_g^* I_1(j\omega) \tag{18}$$

$$2h_\ell a_2 = V_2(j\omega) - Z_\ell I_2(j\omega) \tag{19}$$

$$2h_\ell^* b_2 = V_2(j\omega) + Z_\ell^* I_2(j\omega) \tag{20}$$

where

$$2h_g h_g^* = Z_g + Z_g^* \tag{21}$$

$$2h_\ell h_\ell^* = Z_\ell + Z_\ell^*. \tag{22}$$

Denote $p = \alpha + j\omega$. $h_g(p)$, $h_g^*(p)$, $h_\ell(p)$, $h_\ell^*(p)$, and $h_g(p)/h_g^*(p)$, $h_\ell(p)/h_\ell^*(p)$ should be bounded and analytic at $\Re(p) > 0$. With this definition, there exists the following linear equation for S-parameters:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{23}$$

The matrix S satisfies the unitary condition, and the power gain is given by $G(\omega^2) = |S_{12}(j\omega)|^2$. Obviously, at $x = 0$, we have

$$\phi(0) = \frac{1}{\sqrt{Z_0(0)}}\left(\frac{Z_g^*}{h_g^*} a_1 + \frac{Z_g}{h_g} b_1\right) \tag{24}$$

$$\psi(0) = \sqrt{Z_0(0)}\left(\frac{a_1}{h_g^*} - \frac{b_1}{h_g}\right) \tag{25}$$

$$\phi'(0) = k(0)\phi(0) - j\omega\psi(0) \tag{26}$$

and at $x = \ell$

$$\phi(\ell) = \frac{1}{\sqrt{Z_0(\ell)}}\left(\frac{Z_\ell^*}{h_\ell^*} a_2 + \frac{Z_\ell}{h_\ell} b_2\right) \tag{27}$$

$$\psi(\ell) = \sqrt{Z_0(\ell)}\left(-\frac{a_2}{h_\ell^*} + \frac{b_2}{h_\ell}\right) \tag{28}$$

$$\phi'(\ell) = k(\ell)\phi(\ell) - j\omega\psi(\ell). \tag{29}$$

For a matched NTL, assume $a_1 = 1$, then $b_1 = S_{11}(j\omega)$, $b_2 = S_{12}(j\omega)$, and $a_2 = 0$. We can obtain the following boundary conditions:

$$\phi(0) = \frac{1}{\sqrt{Z_0(0)}}\left(\frac{Z_g^*}{h_g^*} + \frac{Z_g}{h_g} S_{11}\right) \tag{30}$$

$$\psi(0) = \sqrt{Z_0(0)}\left(\frac{1}{h_g^*} - \frac{S_{11}}{h_g}\right) \tag{31}$$

$$\phi(\ell) = \frac{1}{\sqrt{Z_0(\ell)}}\frac{Z_\ell}{h_\ell} S_{12} \tag{32}$$

$$\psi(\ell) = \sqrt{Z_0(\ell)}\frac{S_{12}}{h_\ell}. \tag{33}$$

Let $y_1(x, \omega)$, $y_2(x, \omega)$ be the two independent solutions of (9) with

$$y_1(0, \omega) = 1 \quad y_1'(0, \omega) = 0$$

$$y_2(0, \omega) = 0 \quad y_2'(0, \omega) = 1.$$

The solution of (9) can then be written as

$$\phi(x, j\omega) = \phi(0)y_1(x, \omega) + \phi'(0)y_2(x, \omega). \tag{34}$$

For a lossless line, $q(x)$ and $k(x)$ are real functions, and thus, are y_1 and y_2. Hence,

$$\phi^*(x, j\omega) = \phi^*(0)y_1(x, \omega) + \phi'^*(0)y_2(x, \omega) \tag{35}$$

and together with (34), we obtain

$$y_1(\ell) = \frac{1}{D}\left[\phi(\ell)\phi'^*(0) - \phi^*(\ell)\phi'(0)\right] \tag{36}$$

$$y_2(\ell) = \frac{1}{D}\left[\phi(0)\phi^*(\ell) - \phi^*(0)\phi(\ell)\right]. \tag{37}$$

Similarly, from

$$\phi'(x, j\omega) = \phi(0)y_1'(x, j\omega) + \phi'(0)y_2'(x, j\omega) \tag{38}$$

and its conjugate equation, we have

$$y_1'(\ell) = \frac{1}{D}\left[\phi'(\ell)\phi'^*(0) - \phi'^*(\ell)\phi'(0)\right] \tag{39}$$

$$y_2'(\ell) = \frac{1}{D}\left[\phi(0)\phi'^*(\ell) - \phi^*(0)\phi'(\ell)\right] \tag{40}$$

where the determinant $D = \phi(0)\phi'^*(0) - \phi^*(0)\phi'(0)$, which can be further simplified as

$$\begin{aligned} D &= j\omega\left[\phi(0)\psi^*(0) + \phi^*(0)\psi(0)\right] \\ &= 2j\omega\left(1 - |S_{11}|^2\right) = 2j\omega|S_{12}|^2. \end{aligned} \tag{41}$$

$y_2'(\ell)$ and $y_2(\ell)$ can especially be written as

$$\begin{aligned} y_2(\ell) &= \frac{1}{\omega}\,\frac{\Im\left[\phi(0)\phi^*(\ell)\right]}{|S_{12}|^2} \\ &= \frac{1}{\omega\sqrt{Z_0(0)Z_0(\ell)}}\Im\left(\frac{Z_\ell^*}{h_\ell^*}\,\frac{\frac{Z_g^*}{h_g^*} + \frac{Z_g}{h_g}S_{11}}{S_{12}}\right) \end{aligned} \tag{42}$$

$$\begin{aligned} y_2'(\ell) - k(\ell)y_2(\ell) &= \frac{\Re\left[\phi(0)\psi^*(\ell)\right]}{|S_{12}|^2} \\ &= \sqrt{\frac{Z_0(\ell)}{Z_0(0)}}\,\Re\left(\frac{1}{h_\ell^*}\,\frac{\frac{Z_g^*}{h_g^*} + \frac{Z_g}{h_g}S_{11}}{S_{12}}\right) \end{aligned} \tag{43}$$

$\Re$ and $\Im$ stand for the real and imaginary part, respectively. Thus, to synthesize an NTL means to solve the following inverse problem: for given Z_g, Z_ℓ, and S-parameters, find a proper potential function $q(x)$ such that (9) and boundary conditions (42) and (43) are observed, and $Z_0(x)$ computed from $q(x)$ is practically realizable.

Equations (42) and (43) are a special case when $h = \infty$, and $H_1 = \infty$, $H_2 = -k(\ell)$ in *Theorem 1*. The related μ_j and λ_j must have the following asymptotic estimates [assuming $q(x) \in L_2(0, \ell)$] [13]:

$$\sqrt{\mu_j} = \frac{j\pi}{\ell} + \frac{a_0}{j} + O\left(\frac{1}{j^2}\right) \tag{44}$$

$$\sqrt{\lambda_j} = \frac{(j-0.5)\pi}{\ell} + \frac{b_0}{j-0.5} + O\left(\frac{1}{j^2}\right) \tag{45}$$

where

$$a_0 = \frac{1}{2}\int_0^\ell q(s)\,ds \quad b_0 = \frac{1}{2}\int_0^\ell q(s)\,ds - k(\ell).$$

Clearly, $k(\ell)$ and $(1/2)\int_0^\ell q(s)\,ds$ are definitely determined by the asymptotic characteristics of λ_j and μ_j. As discussed by Levitan [12], if we are only interested in limited frequency intervals, we actually can choose $k(\ell)$ and $\int_0^\ell q(s)\,ds$ for computational convenience, reasonably having the possibility that, at infinity, the estimates still hold.

Wohlers [8] has checked the realizability of an NTL (for real Z_g), and provided several conditions for Z_ℓ and S-parameters under which there exists a proper NTL to realize the required S for a specified Z_ℓ. In this paper, we will derive the required conditions for them in a slightly implicit way, which leads to *Theorem 2*. We will see that this theorem provides a clue for a practical algorithm.

Theorem 2: For given Z_g, Z_ℓ, and S, if they satisfy the condition that the right-hand sides of (42) and (43) are all entire functions of order $1/2$, and have the following asymptotic behavior:

$$\frac{1}{\omega\sqrt{Z_0(0)Z_0(\ell)}}\Im\left(\frac{Z_\ell^*}{h_\ell^*}\,\frac{\frac{Z_g^*}{h_g^*} + \frac{Z_g}{h_g}S_{11}}{S_{12}}\right) \to \frac{\sin\left(\sqrt{\lambda}\,\ell\right)}{\sqrt{\lambda}}, \quad \lambda \to \infty \tag{46}$$

$$\sqrt{\frac{Z_0(\ell)}{Z_0(0)}}\,\Re\left(\frac{\frac{Z_g^*}{h_g^*} + \frac{Z_g}{h_g}S_{11}}{h_\ell^* S_{12}}\right) \to \cos(\sqrt{\lambda}\ell) + b_0\frac{\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}}, \qquad \lambda \to \infty \tag{47}$$

their zeros μ_j and λ_j satisfy (44) and (45) simultaneously. There then exists an NTL with length ℓ, which bears the given S parameters with termination of Z_g and Z_ℓ. The opposite of this theorem is also true.

To prove this theorem, it is sufficient to note that, for a continuous potential $q(x) (0 \leq x \leq \ell)$, $y_2(\ell, \lambda)$ and $y_2'(\ell, \lambda) + Hy_2(\ell, \lambda)$ are all entire functions of order $1/2$ and it is an overdetermined inverse problem when λ_j and μ_j, which satisfy (44) and (45), are known [13].

After this, we will use a terminology of "qualified S, Z_g, and Z_ℓ," if S, Z_g, and Z_ℓ satisfy the conditions in *Theorem 2*.

Remark: If a provided S is not qualified, we can realize it only in a limited frequency range to a certain extent that we are really concerned about in practical problems. What we should do at first is to find a proper approximation of S that satisfy the conditions in *Theorem 2*. Thus, we have changed the problem of NTL synthesis to a problem of approximation of a function in terms of entire functions of order 1/2.

IV. ALGORITHM

A. Algorithm for Qualified S

From a qualified S-parameter, we construct the potential function $q(x)$ by the following algorithm:

Algorithm 1—For Qualified S-Parameter:

Step 1) Take the zeros of the right-hand sides of (42) and (43) as the eigenvalue sequences μ_j and λ_j. From their estimates (44) and (45), calculate $(1/2)\int_0^\ell q(s)\,ds$ and $k(\ell)$.

Step 2) Determine $Z_0(\ell)$ from the asymptotic behavior (46) and (47).

Step 3) There exist a set of integral transformations [14] that map $y_2(x, \omega)$ to a frequency-independent function $H(x, t)$, which can be recovered from necessary eigenvalues

$$y_2(x, \omega) = \frac{\sin(\omega x)}{\omega} + \int_0^x H(x, t)\frac{\sin(\omega t)}{\omega}\,dt \tag{48}$$

$$H_{tt} - H_{xx} + q(x)H = 0 \tag{49}$$

$$H(x, x) = \frac{1}{2}\int_0^x q(s)\,ds. \tag{50}$$

Assume that

$$H(\ell, t) = \sum_{n=1}^{\infty} a_n u_n(t) \cong \sum_{n=1}^{M} a_n u_n(t) \tag{51}$$

$$H_x(\ell, t) - k(\ell)H(\ell, t) = \sum_{n=1}^{\infty} b_n v_n(t) \cong \sum_{n=1}^{M} b_n v_n(t) \tag{52}$$

from the following set of equations:

$$\sum_{n=1}^{M} a_n \int_0^\ell u_n(t)\frac{\sin\left(\sqrt{\mu_m}t\right)}{\sqrt{\mu_m}}\,dt = -\frac{\sin\left(\sqrt{\mu_m}\ell\right)}{\sqrt{\mu_m}} \tag{53}$$

$$\sum_{n=1}^{M} b_n \int_0^\ell v_n(t)\frac{\sin\left(\sqrt{\lambda_m}t\right)}{\sqrt{\lambda_m}}\,dt = -\cos\left(\sqrt{\lambda_m}\ell\right) - b_0\frac{\sin\left(\sqrt{\lambda_m}\ell\right)}{\sqrt{\lambda_m}}, \quad m = 1, \ldots, M. \tag{54}$$

$H(\ell, t)$, $H_x(\ell, t) - k(\ell)H(\ell, t)$ can be calculated, hence, $H_x(\ell, t)$. By the algorithm described in [14], (49) can be solved from these boundary data and $q(x)$ can be constructed from (50). Here, we may choose

$$u_n(t) = \sin\frac{n\pi t}{\ell}$$

and

$$v_n(t) = \sin\frac{(n-0.5)\pi t}{\ell}$$

as the expansion basis functions.

Step 4) With $Z_0(\ell)$ and $k(\ell)$, compute $Z_0(x)$ from (11).

This procedure can be applied only when S_{11} is correctly provided. As was proposed in [8], $S_{11}(p)$ must be a meromorphic function, and $S_{11}(\overline{p}) = \overline{S_{11}(p)}$. $S_{12}(p)$ can be derived from $S_{11}(p)$ by factorizing $1 - S_{11}(p)S_{11}(-p)$.

B. Algorithm for Unqualified S

Consider a special case where Z_g and Z_ℓ are real. In this case, (42) and (43) come to

$$y_2(\ell) = \frac{1}{\omega}\sqrt{\frac{Z_g Z_\ell}{Z_0(0)Z_0(\ell)}}\,\Im\left(\frac{1+S_{11}}{S_{12}}\right) \tag{55}$$

$$y_2'(\ell) - k(\ell)y_2(\ell) = \sqrt{\frac{Z_g Z_0(\ell)}{Z_0(0)Z_\ell}}\,\Re\left(\frac{1+S_{11}}{S_{12}}\right). \tag{56}$$

Denote

$$B_1 = \sqrt{\frac{Z_g Z_\ell}{Z_0(0)Z_0(\ell)}}$$

and

$$B_2 = \sqrt{\frac{Z_g Z_0(\ell)}{Z_0(0)Z_\ell}}$$

for convenience. Assume that $S_{11} = s_{11}e^{-j\theta_{11}}$ is provided, then S_{12} is also determined for a certain NTL. If we write $S_{12} = s_{12}e^{-j\theta_{12}}$, we have $s_{12}^2 = 1 - s_{11}^2$, and θ_{12} should only be chosen such that the conditions in *Theorem 2* are observed. From this point-of-view, we compose the following algorithm, and avoid the factorization proposed by Wohlers.

Algorithm 2—For Unqualified S-Parameter:

Step 1) As $y_2(\ell)$ and $y_2'(\ell) - k(\ell)y_2(\ell)$ are entire functions of order 1/2, we can write

$$f_1(\lambda) = y_2(\ell, \lambda) = \alpha_1 \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{\mu_j}\right) \tag{57}$$

$$f_2(\lambda) = y_2'(\ell, \lambda) - k(\ell)y_2(\ell, \lambda) = \alpha_2 \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_j}\right). \tag{58}$$

Choose $k(\ell) \approx 0$ and $\int_0^\ell q(s)\,ds = 0$, then α_1 and α_2 are determined from (46) and (47) as follows:

$$\alpha_1 = \ell \prod_{j=1}^{\infty} \left(\frac{\mu_j}{Q_j}\right) \tag{59}$$

$$\alpha_2 = \prod_{j=1}^{\infty} \left(\frac{\lambda_j}{P_j}\right) \tag{60}$$

where $\sqrt{Q_j} = j\pi/\ell$ and $\sqrt{P_j} = (j-0.5)\pi/\ell$. We do not use (57) and (58) directly in our computation, but the following alternative ones:

$$f_1(\lambda) = \frac{\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}} \prod_{j=1}^{M} \frac{\mu_j - \lambda}{Q_j - \lambda} \tag{61}$$

$$f_2(\lambda) = \cos\left(\sqrt{\lambda}\ell\right) \prod_{j=1}^{M} \frac{\lambda_j - \lambda}{P_j - \lambda}. \tag{62}$$

Step 2) Let the right-hand sides of (55) and (56) be denoted by

$$G_1(\lambda) = B_1 \frac{g(\lambda)}{\sqrt{\lambda}} \sin\theta \tag{63}$$

$$G_2(\lambda) = B_2 g(\lambda) \cos\theta \tag{64}$$

where $g(\lambda) = |(1+S_{11})/S_{12}|$ is dependent only on S_{11}. $\theta = \arg((1+S_{11})/S_{12})$ and $\theta \to \omega\ell$ when $\omega \to \infty$.

Next, we want to use $f_1(\lambda)$, $f_2(\lambda)$ to approximate $G_1(\lambda)$, $G_2(\lambda)$ at fixed points $[x_i, G_1(x_i)]$ and $[y_i, G_2(y_i)]$, respectively, where

$$-\infty \le x_1 \le \mu_1 \le x_2 \le \mu_2 \le \cdots x_i \le \mu_i \le x_{i+1} \ldots \tag{65}$$

$$-\infty \le y_1 \le \lambda_1 \le y_2 \le \lambda_2 \le \ldots y_i \le \lambda_i \le y_{i+1} \cdots \tag{66}$$

and $G_1(x_i)$, $G_2(y_i)$ satisfy (63) and (64). Here, we choose $x_i = \lambda_i$, $y_i = \mu_i$, and $y_0 = 0$. At $\lambda = \mu_i$, $G_1(\mu_i) = 0$, $G_2(\mu_i) = \pm B_2 g(\mu_i)$, and at $\lambda = \lambda_i$, $G_2(\lambda_i) = 0$, $G_1(\lambda_i) = \pm B_1 g(\lambda_i)/\sqrt{\lambda_i}$. On the other hand, λ_i and μ_i are interlaced [13, Th. 3.4.1]. This means that $f_1(\lambda_i)$ and $f_2(\mu_i)$ must change their signs alternatively; thus, the relating values of the right-hand sides of (55) and (56) also must. According to the estimates (44) and (45), we have the following *sampling* values:

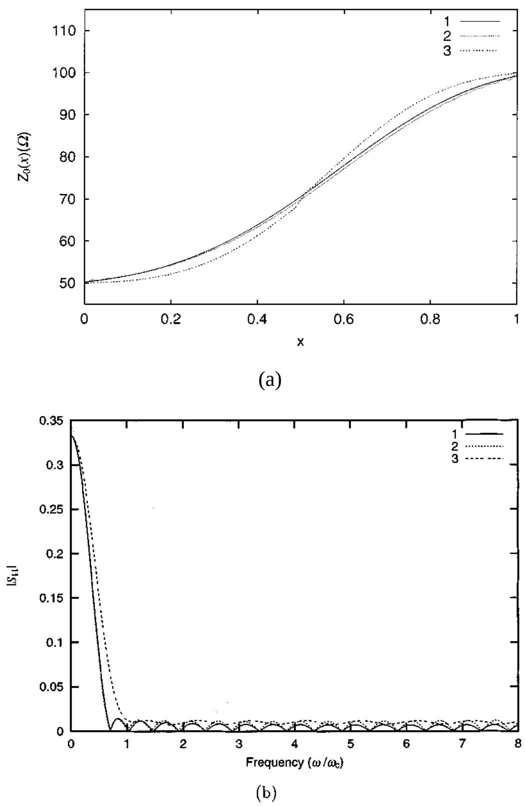
$$G_1(\lambda_i) = (-1)^i B_1 g(\lambda_i)/\sqrt{\lambda_i} \tag{67}$$

$$G_2(\mu_i) = (-1)^{i+1} B_2 g(\mu_i) \tag{68}$$

and additionally $G_2(0) = B_2 g(0)$. In this paper, λ_i and μ_i will be determined by minimizing the following object function:

$$E_R[\lambda_i, \mu_i, B_1, B_2] = \sum_{i=1}^{M} \left[f_1(\lambda_i) - G_1(\lambda_i)\right]^2 + \sum_{i=0}^{M} \left[f_2(\mu_i) - G_2(\mu_i)\right]^2 \tag{69}$$

where $\mu_0 = 0$, and M corresponds to the upper-bound of frequency of being practically concerned with. Under this approximation, S_{11} and S_{12} are fitted only at the points λ_i and μ_i, while they are properly interpolated at other points.



(b)

Fig. 2. (a) Characteristic impedances. (b) Simulated results for $|S_{11}|$. NTL tapers with $Z_g = 50\,\Omega$, $Z_l = 100\,\Omega$. $l = 1$. 1: Dolph–Chebyshev taper $A_r = 0.01$. 2: Equiripple taper $A_r = 0.01$, $A_b = 0$. Equiripple taper 3: $A_r = 0.002$, $A_b = 0.01$.

TABLE I
FIRST TEN OPTIMIZED EIGENVALUES

No.	λ_i	μ_i
1	0.09322025	0.02314400
2	0.36951025	0.20675260
3	0.80541548	0.56158928
4	1.43666031	1.01080555
5	2.46296384	2.32045748
6	3.57011573	2.69370521
7	4.84246483	4.72918638
8	6.82492401	5.62769745
9	8.29679006	7.35697636
10	10.0989503	9.08134626

Remark: Practically, as we are only concerned about a limited frequency interval, we can always choose $k(\ell) \approx 0$ and $\int_0^\ell q(s)\,ds = 0$. We can choose the initial data $\lambda_{i0} = P_i$ and $\mu_{i0} = Q_i$, or take $\theta_{12} = \theta_{11} = \omega\ell$, and choose λ_{i0}, μ_{i0} as the zeros of $\Re((1+S_{11})/S_{12})$ and $\Im((1+S_{11})/S_{12})$, respectively.

It is important to point out that for an NTL with limited length ℓ, the ripples in both passband and stopband are all intrinsic, and are correlated with the eigenvalues λ_j and μ_j. When ℓ becomes larger, λ_j and μ_j will come closer, and the ripples tend to become lower.

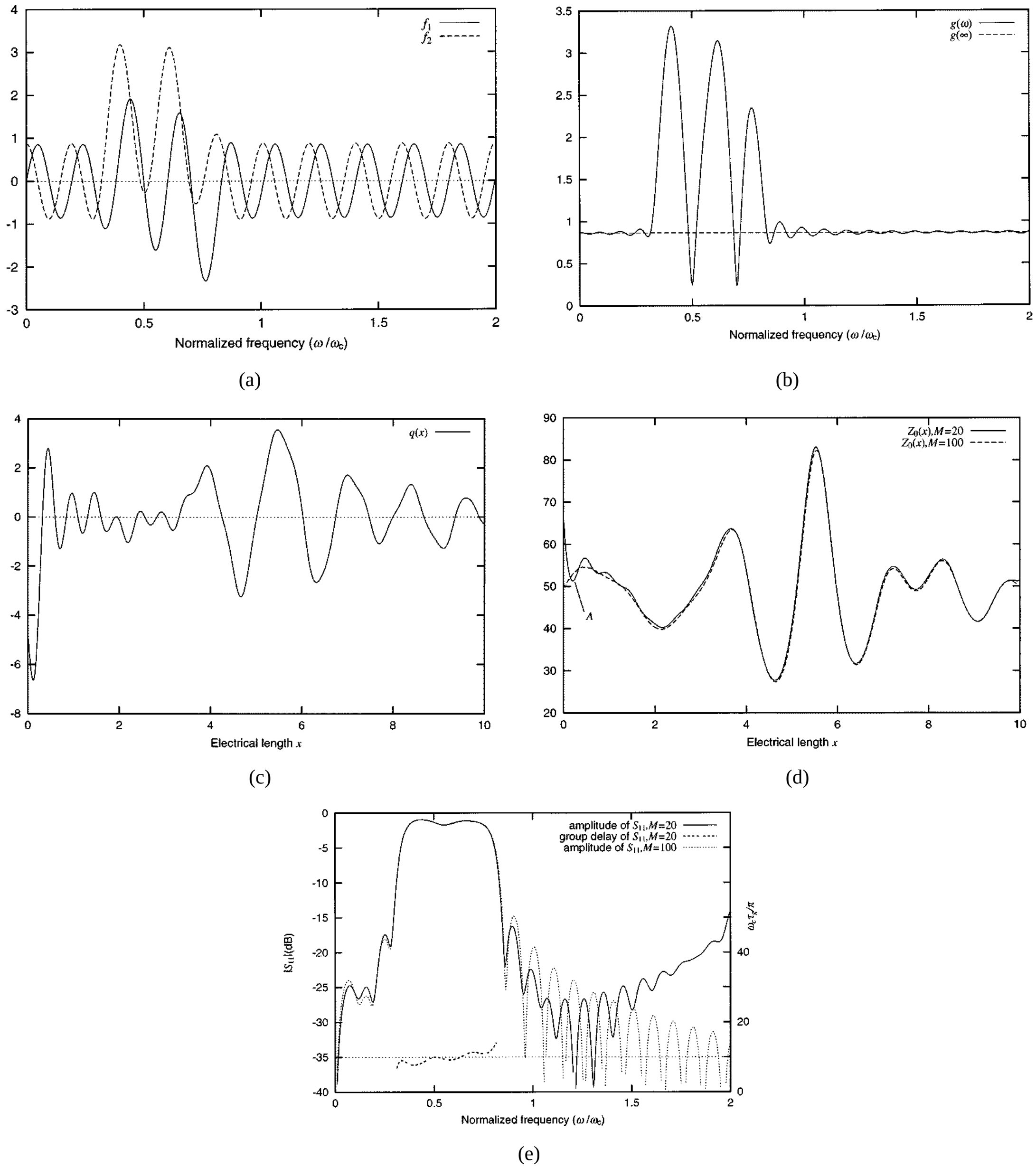


Fig. 3. (a) Optimized $f_1 = \omega y_2(l, \omega)$, $f_2 = \omega y_2'(\lambda)$, $B_1 = 0.875$, $B_2 = 0.853$. (b) Approximated $g(\omega)$, $q(\infty) = 1/\sqrt{B_1^2 + B_2^2}$. (c) Constructed $q(x)$. (d) Related $Z_0(x)$ with $Z_g = Z_l = 50\,\Omega$. Optimized $Z_0(0) = 67\,\Omega$, $Z_0(l) = 51\,\Omega$ when $M = 20$, $Z_0(0) = 52\,\Omega$, $Z_0(0) = 50\,\Omega$ when $M = 100$. (e) Simulated results of S_{11}. Synthesis example: $S_{11} = 0.85\exp(-j10\omega)$ for $\omega/\omega_c \in [0.3,\ 0.8]$. $M = 20$ and 100. $E_R \leq 10^{-10}$. M: the number of λ (or μ_j) to be modified in the optimization process).

C. Examples

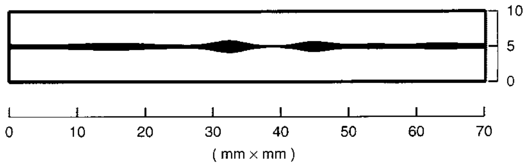
Example 1—Equiripple Tapers: We use the above algorithm to design equiripple tapers. Assume that $S_{11}(j\omega) = A_b + A_r \cos(\sqrt{\omega^2 - A_0}) \exp(-j\omega\ell)$ when $\omega \leq \omega_0 < \infty$, where A_r is ripple level and A_b is a constant. $S_{11}(0) = A_b + A_r \cosh(\sqrt{A_0})$ and $S_{11}(0) = (Z_\ell - Z_g)/(Z_\ell + Z_g)$. The designed $Z_0(x)$ of two kinds of equiripple tapers and their simulated $|S_{11}(j\omega)|$ are shown in Fig. 2. The results are compared with those of a Dolph–Chebyshev taper.

This example shows that, by adjusting the eigenvalue μ_i and λ_i, we can modify local frequency response to a certain extent with resolution dependent upon ℓ. A similar method to control taper ripples can be found in [15].

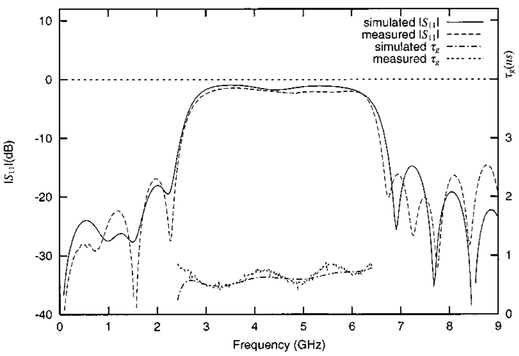
Example 2—NTL Passband Filter: Suppose that $s_{11} = 0.85$ when the normalized frequency $\omega/\omega_c \in [0.3,\ 0.8]$ and $s_{11} = 0$ through out this interval. $\theta_{11} = (\pi\omega/\omega_c)\tau_g$, and $Z_g = Z_\ell = 50\ \Omega$. Here, ω_c is a normalizing frequency and τ_g is a group delay.

We choose $\ell = \tau_g\omega_c/\pi = 10$, which means we select the length of NTL to be ten half-wavelengths at ω_c. Thus, there are ten eigenvalues for λ_i and μ_i in the interval of $\omega/\omega_c \in [0,\ 1]$. Assume that we only mind the stopband up to $\omega/\omega_c = 2$. Therefore, we should select $M \geq 20$, where M is the eigenvalue number that is necessary to be optimized [see (69)]. Let $k(\ell) = 0$ and $\int_0^\ell q(s)\,ds = 0$.

Table I gives the first ten optimized λ_i and μ_i. Fig. 3(a) and (b) shows the approximated $y_2(\lambda)$, $y_2'(\lambda)$, and $g(\lambda)$, while Fig. 3(c)



(a)



(b)

Fig. 4. (a) Device pattern. (b) Simulated and measured result of S_{11} device characteristics. f_c: 8 GHz, passband: 2.4–6.4 GHz. M: 100, device size: 1 cm × 7.02 cm, measured through an HP8720C network analyzer.

and (d) shows the results of $q(x)$ and $Z_0(x)$, respectively. We also present a simulated result in Fig. 3(e), which is obtained by treating the NTL as cascaded steps.

As is shown in Fig. 3(d), there exists an abnormal variation of $Z_0(x)$ near $x = 0$. This is due to the choice of $k(\ell) = 0$ and, in the mean time, M is relatively small. We have repeated the similar process with $M = 100$. The results are also shown in Fig. 3(d) and (e). It can be seen that the abnormal segment is diminished, and only the stopband ripples are affected. Actually, we also found that if we simply cut that abnormal segment off [until point A, see Fig. 3(d)], the affection on S_{11} is negligible.

We have tested an experimental device. It was built on an RT/duroid 6010.5 printed circuit board (PCB) with a copper conductor ($\varepsilon_r = 10.2$, thickness $H = 0.635$ mm). The normalizing frequency is $f_c = \omega_c/2\pi = 8$ GHz, the passband is 2.4–6.4 GHz. The device was produced utilizing a circuit board plotter (LPKF ProtoMat 91s/VS), as we intend only to demonstrate the principle. Fig. 4(a) shows the device pattern and Fig. 4(b) shows the simulated and measured results of S_{11}. The errors are partly due to the over milling of the PCB and mismatching at the ends of the NTL. The passband characteristics will certainly be improved if we choose larger ℓ (longer NTL). This is obvious, because for larger ℓ, λ_i and μ_i become smaller, thus, there will be more sampling points in the passband.

Obviously, when we add a directional coupler at input port, this NTL is actually a bandpass filter.

V. Conclusions

Proper boundary conditions from S-parameters are presented. The synthesis of an NTL is turned into an inverse classical SLP. The examples illustrate that this method can be used to design NTL filters and transducers for almost arbitrarily provided source and load impedances. If $S_{11}(j\omega)$ can be properly provided, this algorithm can also be applied to frequency-dependent Z_ℓ and Z_g.

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